

ON FORMAL SCHUBERT POLYNOMIALS

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ABSTRACT. Present notes can be viewed as an attempt to extend the notion of Schubert/Grothendieck polynomial of Lascoux-Schützenberger to the context of an arbitrary formal group law and of an arbitrary oriented cohomology theory.

Let $F \in R[[x, y]]$ be a commutative one-dimensional formal group law over a commutative unital ring R and let \mathbf{h} be an algebraic oriented cohomology theory with the coefficient ring $\mathbf{h}(pt) = R$. According to Levine-Morel [LM07] there is a 1-1 correspondence between such F 's and universal \mathbf{h} 's. Indeed, given an oriented theory \mathbf{h} the respective formal group law is determined by the Quillen formula for the first charactersitic classes in the theory \mathbf{h} of the tensor product of line bundles

$$c_1^{\mathbf{h}}(L_1 \otimes L_2) = F(c_1^{\mathbf{h}}(L_1), c_1^{\mathbf{h}}(L_2))$$

and given a formal group law F over R one obtains the respective universal theory \mathbf{h} by tensoring with the algebraic cobordism Ω , i.e.

$$\mathbf{h}(-) := \Omega(-) \otimes_{\Omega(pt)} R,$$

where $\Omega(pt) \rightarrow R$ is obtained by specializing coefficients of the universal formal group law.

For example, additive formal group law $F_a(x, y) = x + y$ corresponds to the Chow theory $\mathbf{h} = CH$, multiplicative $F_m(x, y) = x + y - xy$ to the Grothendieck $\mathbf{h} = K_0$ and the universal one F_u to the algebraic cobordism $\mathbf{h} = \Omega$. Observe that in the first two cases $\mathbf{h}(pt) = \mathbb{Z}$ and in the last case the coefficient ring is the Lazard ring which is infinitely generated over \mathbb{Z} .

Let G be a split semisimple linear group over a field k containing a split maximal torus T . Following [CPZ, §2] consider the formal group algebra

$$R[[M]]_F := R[[x_\omega]]_{\omega \in M} / (x_0, x_{\omega+\omega'} - F(x_\omega, x_{\omega'})),$$

where M is the weight lattice, together with the augmentation map $\epsilon: R[[M]]_F \rightarrow R$, $x_\omega \mapsto 0$. From the geometric point of view $R[[M]]_F$ models the completion of the equivariant cohomology $\mathbf{h}_T(pt)$ and the map ϵ is the forgetful map. Algebraically, $R[[M]]_F$ is non-canonically isomorphic to the ring of formal power series in $rk(M)$ variables.

Consider the algebra of formal divided difference operators $\mathcal{D}(M)_F$ on $R[[M]]_F$ and let $\epsilon\mathcal{D}(M)_F^* := \text{Hom}_R(\epsilon \circ \mathcal{D}(M)_F, R)$ denote the dual of the algebra of augmented operators. The main result of [CPZ, §13] says that if \mathbf{h} is a (weakly birationally invariant) oriented cohomology theory corresponding to F (e.g. $\mathbf{h} = CH$,

Key words and phrases. Hecke algebra, elliptic formal group law, Kazhdan-Lusztig basis, Bott-Samelson resolution, Schubert polynomial, Grothendieck polynomial.

K_0 or Ω), then there is an R -algebra isomorphism

$$(1) \quad \epsilon\mathcal{D}(M)_F^* \simeq \mathfrak{h}(X),$$

where X is the variety of Borel subgroups of G . Moreover, it was shown that the R -basis of $\mathfrak{h}(X)$ consisting of classes of Bott-Samelson resolutions of Schubert varieties corresponds to the basis of $\epsilon\mathcal{D}(M)_F^*$ constructed as follows:

First, for each element of the Weyl group $w \in W$ one chooses a reduced decomposition $w = s_{i_1} s_{i_2} \dots s_{i_r}$ into a product of simple reflections and denotes by $I_w = (i_1, \dots, i_r)$ the respective reduced word. Then one shows that the R -linear operators $\epsilon C_{I_w}^F$ defined by composing ϵ with the composite of the formal divided difference operators $C_{I_w}^F = C_{i_1}^F \circ \dots \circ C_{i_r}^F$ form a basis of $\epsilon\mathcal{D}(M)_F$ [CPZ, Prop. 5.4]. Finally, the elements $A_{I_w}(z_0)$ give the desired basis, where z_0 is the element of $\epsilon\mathcal{D}(M)_F^*$ dual to $\epsilon C_{I_{w_0}}^F(u_0)$ for some specially chosen $u_0 \in (\ker \epsilon)^{\dim X}$, w_0 is the element of maximal length and A_i is the operator on $\epsilon\mathcal{D}(M)_F^*$ dual to the operator on $\epsilon\mathcal{D}(M)_F$ given by composition on the right with C_i^F [CPZ, Thm. 13.13].

One of the major difficulties in extending the Schubert calculus to such generalized theories \mathfrak{h} (and, hence, the Schubert polynomials) is the fact that all mentioned bases are non-canonical, i.e. depend on choices of reduced decompositions. Moreover, according to [BE90] they are canonical if and only if F has the form $F(u, v) = u + v - \beta uv$ for some $\beta \in R$. In other words, the Bott-Samelson resolutions of Schubert varieties provide a canonical basis of $\mathfrak{h}(X)$ only for Chow groups ($\beta = 0$), Grothendieck K_0 (β is invertible) and connective K -theory ($\beta \neq 0$ is non-invertible).

In the present notes we try to overcome this difficulty and, hence, provide a *canonical basis* of $\mathfrak{h}(X)$ by either

- (1) averaging over all reduced decompositions, i.e. over all classes of Bott-Samelson resolutions; or
- (2) exploiting the Kazhdan-Lusztig theory in the case of a special elliptic formal group law.

Observe that approach (1) works only after inverting the Hurwitz numbers, e.g. over \mathbb{Q} , but over \mathbb{Q} all formal group laws become isomorphic. Therefore, one may suspect that we simply reduce to the known cases of additive or multiplicative formal group laws. But this isomorphism does not preserve the formal difference operators as well as many other structures, so this is not the case.

Approach (2) seems to be even more interesting as it gives a canonical basis integrally. However, we don't know how to extend it to other examples of formal group laws.

1. AVERAGING OVER REDUCED EXPRESSIONS

Consider the evaluation map $\mathfrak{c}^F: R[[M]]_F \rightarrow \epsilon\mathcal{D}(M)_F^*$ of [CPZ, §6]. Observe that on the level of cohomology (after identifying with $\mathfrak{h}(X)$) it coincides with the characteristic map induced by $\omega \mapsto c_1^{\mathfrak{h}}(L(\omega))$. In the case of the additive or multiplicative formal group law it gives the characteristic map described by Demazure in [De74], [De73]. Moreover, according to [CPZ, Thm. 6.9] if the Grothendieck torsion index τ of G is invertible (this always holds for Dynkin types A and C), then the kernel of \mathfrak{c}^F is the ideal \mathcal{I}_F^W generated by augmented W -invariant elements, and we

obtain an R -algebra isomorphism

$$(2) \quad R[[M]]_F/\mathcal{I}_F^W \simeq \epsilon\mathcal{D}(M)_F^*,$$

which in view of the results of [HMSZ] and [CZZ] relate the invariant theory of W with an F -version of the Hecke ring of Kostant-Kumar [KK86], [KK90], [Ku02] and Bressler-Evens [BE87]. In general, though the kernel of \mathfrak{c}^F always contains the invariant ideal \mathcal{I}^W , the induced map $\mathfrak{c}^F: R[[M]]_F/\mathcal{I}_F^W \rightarrow \epsilon\mathcal{D}(M)_F^*$ is neither injective nor surjective.

Observe also that if τ is invertible, then we can identify the basis $A_{I_w}(z_0)$ of $\epsilon\mathcal{D}(M)_F^*$ with the basis $C_{I_w^{rev}}(u_0)$ of $R[[M]]_F/\mathcal{I}_F^W$, where $u_0 = [pt]$ corresponds to the class of a point (see [CPZ, Thm. 6.7]). The latter suggest to define for each $w \in W$ the following element in $R_{\mathbb{Q}}[[M]]_F/\mathcal{I}_F^W$:

$$(3) \quad P_w^F := \frac{1}{|\text{red}(w)|} \sum_{I_w \in \text{red}(w)} C_{I_w^{rev}}([pt]),$$

where the sum is taken over the set $\text{red}(w)$ of all reduced words of w .

The results of [HMSZ] and [CZZ] then imply that $\{P_w^F\}_{w \in W}$ is the desired canonical basis over \mathbb{Q} :

Theorem 1. *The elements $\{P_w^F\}_{w \in W}$ form a $R_{\mathbb{Q}} = R \otimes_{\mathbb{Z}} \mathbb{Q}$ -basis of $R[[M]]_F/\mathcal{I}_F^W$ and, hence, of $\mathfrak{h}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. By [HMSZ, Prop. 5.8] and [CZZ, Lem. 7.1] the difference $(C_i C_j)^{\text{om}_{ij}} - (C_j C_i)^{\text{om}_{ij}}$ is a linear combination of terms of length strictly smaller than $2m_{ij}$ (here m_{ij} is the exponent in the respective Coxeter relation). Therefore, each P_w^F can be written as $P_w^F = (C_{I_w^{rev}} + (\text{products of smaller length}))([pt])$. So the matrix expressing P_w^F in terms of the usual basis $\{C_{I_w^{rev}}\}_{w \in W}$ corresponding to a fixed choice of reduced decompositions $\{I_w\}_{w \in W}$ is upper-triangular with 1's on the main diagonal. \square

Consider a root system of Dynkin type A_n . Let $\{e_1, \dots, e_{n+1}\}$ be the standard basis with $\alpha_i = e_i - e_{i+1}$ the set of simple roots. Consider a ring homomorphism

$$R[t_1, t_2, \dots, t_{n+1}] \rightarrow R[[\Lambda]]_F, \text{ given by } t_i \mapsto x_{-e_i}.$$

It is S_{n+1} -equivariant, therefore, it induces a map on quotients

$$(4) \quad R[t_1, t_2, \dots, t_{n+1}]/I \rightarrow R[[\Lambda]]_F/\mathcal{I}_F^W,$$

where I is the ideal generated by symmetric functions. By Hornbostel-Kiritchenko [HK, Thm. 2.6] this is an R -algebra isomorphism.

Definition 2. We define an F -Schubert polynomial π_w^F to be the image of P_w^F in the quotient $R[t_1, \dots, t_{n+1}]/I$ via the isomorphism (4).

If F has the form $F(u, v) = u + v - \beta uv$ for some $\beta \in R$, i.e. exactly the formal group law for which the respective composites C_{I_w} do not depend on choices of reduced words of w , then for $\beta = 0$ (resp. for $\beta = 1$) π_w^F coincide with the respective Schubert (resp. Grothendieck) polynomials of Lascoux-Schützenberger (e.g. see [Fo94], [FK]) and for arbitrary β we obtain polynomials studied in [FK] and [Hu12].

Observe that under this isomorphism the class of the point $[pt]$ corresponds to the class of the polynomial $t_1^n t_2^{n-1} \dots t_n$ and the formal divided difference operator

$C_i(u) = \frac{u}{x_{-i}} + \frac{s_i(u)}{x_i} = (1 + s_i)(\frac{u}{x_{-i}})$ (here $x_{-i} = x_{-\alpha_i} = x_{e_{i+1}-e_i}$) corresponds to the operator

$$(5) \quad C_i(f) = (1 + \sigma_i)(\frac{f}{\rho_i}),$$

where σ_i swaps t_i and t_{i+1} and ρ_i is the formal power series given by $t_i -_F t_{i+1} = F(t_i, \iota(t_{i+1}))$, where $\iota(x)$ is the formal inverse series of x .

By definition, operators C_i are $R[[M]]_F^{W_i}$ -linear (here $W_i = \langle s_i \rangle$) and satisfy [CPZ, Prop. 3.13]:

$$C_i(uv) = C_i(u)v + s_i(u)C_i(v) - \kappa_i s_i(u)v \text{ and } C_i(1) = \kappa_i,$$

where $\kappa_i = \frac{1}{\rho_i} + \frac{1}{\iota(\rho_i)} \in R[[M]]_F$. We also have $C_i(t_j f) = t_j C_i(f)$ for $j \neq i, i+1$ and

$$C_i(t_i) = \frac{t_i}{t_i -_F t_{i+1}} + \frac{t_{i+1}}{t_{i+1} -_F t_i} = \frac{t_i -_F t_{i+1} +_F t_{i+1}}{t_i -_F t_{i+1}} + \frac{t_{i+1}}{t_{i+1} -_F t_i} = \frac{F(\rho_i, t_{i+1}) - t_{i+1}}{\rho_i} + t_{i+1} \kappa_i.$$

Using these formulas one can compute the polynomials π_w^F .

Our goal now is (using these formulas) to express each polynomial π_w^F as a linear combination of sub-monomials of $t_1^n t_2^{n-1} \dots t_n$ with coefficients from R .

Example 3. We can write an arbitrary formal group law F as

$$F(x, y) = x + y - xyg(x, y).$$

This implies that

$$\frac{1}{z} + \frac{1}{\iota(z)} = g(z, \iota(z)).$$

If $F(x, y) = x + y + a_{11}xy + a_{12}xy(x + y) + O(4)$, then

$$g(x, y) = -a_{11} - a_{12}(x + y) - a_{13}(x^2 + y^2) - a_{22}xy + O(3)$$

and

$$\iota(x) = -x + a_{11}x^2 - a_{11}^2x^3 + O(4).$$

So, after substituting, we obtain

$$\begin{aligned} x -_F y &= x + (-y + a_{11}y^2 - a_{11}^2y^3) + a_{11}x(-y + a_{11}y^2) + a_{12}x(-y)(x - y) + O(4) = \\ &= (x - y) - a_{11}y(x - y) + a_{11}^2y^2(x - y) - a_{12}xy(x - y) + O(4), \end{aligned}$$

and, hence,

$$\begin{aligned} (x -_F y) + (y -_F x) &= a_{11}(x - y)^2 - a_{11}^2(x + y)(x - y)^2 + O(4), \\ (x -_F y)^2 + (y -_F x)^2 &= 2(x - y)^2 - 2a_{11}(x + y)(x - y)^2 + O(4), \\ (x -_F y)(y -_F x) &= -(x - y)^2 + a_{11}(x + y)(x - y)^2 + O(4). \end{aligned}$$

Combining these together we get

$$\begin{aligned} g(x -_F y, y -_F x) &= -a_{11} - a_{12}a_{11}(x - y)^2 - 2a_{13}(x - y)^2 + a_{22}(x - y)^2 + O(3) = \\ &= -a_{11} - (a_{11}a_{12} + 2a_{13} - a_{22})(x - y)^2 + O(3). \end{aligned}$$

We then obtain

$$r(x, y) = \frac{F(x, y) - y}{x} = 1 + a_{11}y + a_{12}y(x + y) + a_{13}y(x^2 + y^2) + a_{22}xy^2 + O(4).$$

Hence,

$$\begin{aligned} r(x -_F y, y) &= 1 + a_{11}y + a_{12}y(x - a_{11}y(x - y)) + a_{13}y((x - y)^2 + y^2) + a_{22}(x - y)y^2 + O(4) = \\ &= 1 + a_{11}y + a_{12}xy + (a_{22} - a_{12}a_{11})(x - y)y^2 + a_{13}y((x - y)^2 + y^2) + O(4), \end{aligned}$$

and, therefore,

$$r(x -_F y, y) + yg(x -_F y, y -_F x) = 1 + a_{12}xy + (a_{22} - a_{12}a_{11})xy(x - y) + a_{13}xy(2y - x) + O(4).$$

Observe that there is a relation $2(a_{22} - a_{11}a_{12}) = 3a_{13}$ in the Lazard ring (the only relation in degree 4) which gives

$$2[(a_{22} - a_{12}a_{11})xy(x - y) + a_{13}xy(2y - x)] = 2a_{13}xy(x + y).$$

In particular, if 2 is invertible and $x = t_i$, $y = t_{i+1}$ for the type A_2 , then $t_i t_{i+1}(t_i + t_{i+1})$ is in the ideal I generated by symmetric functions in t_1, t_2, t_3 , meaning that for $i = 1, 2$

$$C_i(t_i) = \frac{t_i}{t_i -_F t_{i+1}} + \frac{t_{i+1}}{t_{i+1} -_F t_i} = r(t_i -_F t_{i+1}, t_{i+1}) + t_{i+1}\kappa_i = 1 + a_{12}t_i t_{i+1}.$$

Example 4. Using the formulas above we obtain the following expressions for π_w^F in the A_2 -case for an arbitrary F (this agrees with computations at the end of [HK] and [CPZ])

$$\pi_1^F = C_1([pt]) = t_1 t_2, \quad \pi_2^F = C_2([pt]) = t_1^2,$$

Indeed,

$$C_1(t_1^2 t_2) = \frac{t_1^2 t_2}{t_1 -_F t_2} + \frac{t_1 t_2^2}{t_2 -_F t_1} = t_1 t_2 C_1(t_1) = t_1 t_2$$

and

$$C_2(t_1^2 t_2) = \frac{t_1^2 t_2}{t_2 -_F t_3} + \frac{t_1^2 t_3}{t_3 -_F t_2} = t_1^2 \left(\frac{t_2}{t_2 -_F t_3} + \frac{t_3}{t_3 -_F t_2} \right) = t_1^2 (1 + a_{12} t_2 t_3) = t_1^2.$$

$$\pi_{21}^F = C_{12}([pt]) = t_1 + t_2 + a_{11}\pi_1^F, \quad \pi_{12}^F = C_{21}([pt]) = C_2(t_1 t_2) = t_1,$$

Indeed,

$$C_1(t_1^2) = C_1(t_1)t_1 + t_2 C_1(t_1) - \kappa_1 t_1 t_2 = (1 + a_{12} t_1 t_2)(t_1 + t_2) + a_{11} t_1 t_2 = t_1 + t_2 + a_{11} t_1 t_2$$

and

$$C_2(t_1 t_2) = t_1 C_2(t_2) = t_1 (1 + a_{12} t_2 t_3) = t_1$$

And for the element of maximal length $w_0 = (121) = (212)$ we obtain

$$C_{212}([pt]) = 1 + a_{12}\pi_2^F, \quad C_{121}([pt]) = C_1(t_1) = 1 + a_{12}t_1 t_2 = 1 + a_{12}\pi_1^F.$$

Indeed,

$$\begin{aligned} C_2(t_1 + t_2 + a_{11}t_1 t_2) &= t_1 C_2(1) + C_2(t_2) + a_{11}t_1 C_2(t_2) = \\ &= t_1(-a_{11} - (a_{11}a_{12} + 2a_{13} - a_{22})(t_2 - t_3)^2) + 1 + a_{12}t_2 t_3 + a_{11}t_1(1 + a_{12}t_2 t_3) = \\ &= 1 + a_{12}t_2 t_3 - \frac{1}{2}a_{13}t_1(t_2 - t_3)^2 = 1 + a_{12}t_2 t_3 = 1 + a_{12}t_1^2 \end{aligned}$$

as $t_1^2 \equiv t_2 t_3$ and $t_1(t_2 - t_3)^2$ is in the ideal $(t_1(t_2 - t_3)^2 \equiv t_1(t_2^2 + t_3^2) \equiv t_1^3 \equiv 0)$.

Therefore,

$$\pi_{w_0}^F = 1 + \frac{1}{2}a_{12}(t_1^2 + t_1 t_2).$$

Observe that the twisted braid relation (which leads to the dependence on choices) of [HMSZ, Prop. 5.8] then coincides with

$$C_{121} - a_{12}C_1 = C_{212} - a_{12}C_2.$$

2. A SPECIAL ELLIPTIC FORMAL GROUP LAW AND THE KAZHDAN-LUSTIG BASIS

Consider an elliptic curve given in Tate coordinates by

$$(1 - \mu_1 x - \mu_2 x^2)y = x^3.$$

The corresponding formal group law over the coefficient ring $R = \mathbb{Z}[\mu_1, \mu_2]$ is given by (e.g. [BB10, Example 63]),

$$F(x, y) := \frac{x+y-\mu_1 xy}{1+\mu_2 xy}$$

and will be called a *special elliptic* formal group law. Observe that by definition, we have

$$F(x, y) = x + y - xy(\mu_1 + \mu_2 F(x, y)), \text{ so } a_{11} = -\mu_1 \text{ and } a_{12} = -\mu_2.$$

By [HMSZ, Theorem 5.14] for the type A_n the algebra $\mathcal{D}(M)_F$ is generated by operators C_i , $i \in 1..n$, and multiplications by elements $u \in R[[M]]_F$ subject to the following relations:

- (a) $C_i^2 = \mu_1 C_i$
- (b) $C_{ij} = C_{ji}$ for $|i - j| > 1$,
- (c) $C_{iji} - C_{jij} = \mu_2(C_j - C_i)$ for $|i - j| = 1$ and
- (d) $C_i u = s_i(u)C_i + \mu_1 u - C_i(u)$,

Recall that the Iwahori-Hecke algebra \mathcal{H} of the symmetric group S_{n+1} is (after the respective normalization) an $\mathbb{Z}[t, t^{-1}]$ -algebra with generators T_i , $i = 1..n$, subject to the following relations:

- (A) $(T_i - t^{-1})(T_i + t) = 0$ or, equivalently, $T_i^2 = (t^{-1} - t)T_i + 1$,
- (B) $T_{ij} = T_{ji}$ for $|i - j| > 1$ and
- (C) $T_{iji} = T_{jij}$ for $|i - j| = 1$.

Observe that T_i 's appearing in the classical definition of the Iwahori-Hecke algebra in [CG10, Def. 7.1.1] correspond to tT_i in our notation, where $t = q^{-1/2}$.

Following [HMSZ, Def. 5.3] let D_F denote the R -subalgebra of $\mathcal{D}(M)_F$ generated by the elements C_i , $i = 1..n$, only. In [HMSZ, Prop. 6.1] it was shown that for $F = F_a$ (resp. $F = F_m$) D_F is isomorphic to the nil-Hecke algebra (resp. the 0-Hecke algebra) of Kostant-Kumar.

Comparing the relations for D_F and \mathcal{H} we see that for $R = \mathbb{Z}[t, t^{-1}][\frac{1}{t+t^{-1}}]$, $\mu_1 = 1$ and $\mu_2 = -\frac{1}{(t+t^{-1})^2}$ there is an isomorphism of R -algebras (see [CZZ1] and [LNZ] for the case of an arbitrary root system)

$$(6) \quad \mathcal{H}[\frac{1}{t+t^{-1}}] \simeq D_F \quad \text{given on generators by } T_i \mapsto (t + t^{-1})C_i - t, \quad i = 1..n.$$

By definition of (6) the involution on \mathcal{H} (sending $t \mapsto t^{-1}$ and $T_i \mapsto T_i^{-1}$) corresponds to the involution on D_F obtained by extending the involution $t \mapsto t^{-1}$ on the coefficient ring. Observe that each push-pull element $C_i = \frac{1}{t+t^{-1}}(T_i + t)$ is invariant under this involution.

Consider the Kazhdan-Lusztig basis $\{C'_w\}_{w \in W}$ on \mathcal{H} (e.g. see [CG10]). Recall that it is unique and *does not depend on choices* of reduced decompositions. After the respective normalization we have

$$C'_w = T_w + \sum_{v < w} t\pi_{v,w}(t)T_v,$$

where $\deg \pi_{v,w} \leq l(w) - l(v) - 1$ and $\pi_{v,w}$ are the Kazhdan-Lusztig polynomials. For instance, $C'_i = T_i + t$, $C'_{ij} = T_{ij} + t(T_i + T_j) + t^2$ and $C'_{iji} = T_{iji} + t(T_{ij} + T_{ji}) + t^2(T_i + T_j) + t^3$.

Let C_w denote the element in D_F that corresponds to C'_w via (6). Choose a reduced word I_w for each $w \in W$. Then

$$C_w = (t + t^{-1})^{l(w)}(C_{I_w} + \text{lower degree terms}) + \sum_{v < w} t\pi_{v,w}(t)(t + t^{-1})^{l(v)}(C_{I_v} + \text{lower degree terms})$$

where the right hand side does not depend on choices of reduced decompositions. This suggests the following

Definition 5. We define the special elliptic polynomial π_w^{se} to be the image in $\mathbb{Z}[t, t^{-1}][t_1, \dots, t_{n+1}]/I$ of the element $\frac{1}{(t+t^{-1})^{l(w)}}C_{w^{-1}}([pt])$ via (4).

We expect polynomials π_w^{se} to play the same role (in the special elliptic case) as the Schubert (resp. Grothendieck) polynomials for Chow groups (resp. K_0).

Example 6. For the type A_2 we obtain

$$\pi_i^{se} = C_i([pt]), \quad \pi_{ij}^{se} = C_j C_i([pt])$$

and for the element of maximal length we obtain exactly the twisted braid relation

$$\pi_{w_0}^{se} = (C_{121} + \mu_2 C_1)([pt]) = (C_{212} + \mu_2 C_2)([pt]) = 1.$$

Remark 7. It would be interesting to see

- (1) that $\pi_{w_0}^{se} = 1$, for the element of maximal length w_0 .
- (2) whether π_w^{se} corresponds to the class of an *actual* resolution of the respective Schubert variety X_w .

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